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J. Phys. A: Math. Gen., 13 (1980) 149–160. Printed in Great Britain

Phase-space projection identities for diffraction catastrophes

M V Berry and F J Wright†
H H Wills Physics Laboratory, Bristol University, Tyndall Avenue, Bristol BS8 1TL, UK

Received 6 March 1979

Abstract. For each standard 'diffraction catastrophe' wavefunction ψ, describing the interference pattern near a stable caustic, we derive two nonlinear identities. These relate the intensity $|\psi|^2$ to an integral over the wavefunction ψ corresponding to the same catastrophe, or to a less singular one. The identities are interpreted in terms of projections of Wigner functions from phase space onto coordinate space.

1. Introduction

Diffraction near a smooth caustic in the plane, or near a smooth caustic surface in space, is described by a function derived by Airy (1838). In modern notation (Abramowitz and Stegun 1964) the Airy function is

$$\mathrm{Ai}(C)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}S\,\exp[\mathrm{i}(\tfrac{1}{3}S^3+CS)], \tag{1}$$

where C is a coordinate measuring distance from the caustic. Ai (C) satisfies a surprising nonlinear relation, obtained by Balazs and Zipfel (1973) and Berry (1977a) by projecting a quantum-mechanical Wigner function from phase space onto coordinate space. This 'projection identity' is

$$\mathrm{Ai}^2\,(C)=\frac{1}{2^{1/3}\pi}\int_{-\infty}^{\infty}\mathrm{d}u\;\mathrm{Ai}\,(2^{2/3}(C+u^2)). \tag{2}$$

Our purpose here is to obtain two series of projection identities which constitute the generalisations of equation (2) for more complicated caustics. The corresponding wavefunctions ψ (generalisations of the Airy function) are the 'diffraction catastrophes', constructed according to a rule described in § 2. Each projection identity relates a wave intensity $|\psi|^2$ to an integral over a wavefunction ψ. The method for generating them is explained in § 2. Explicit results for cuspoid and umbilic caustics are presented in § 3 and § 4 respectively. A quantum-mechanical argument, showing how the identities can be interpreted in terms of wavefunctions and Wigner functions associated with a smooth manifold in phase space, is given in § 5. Their detailed implications for Wigner functions are discussed in § 6.

† Present address: Department of Applied Mathematics, Queen Mary College, Mile End Road, London E1 4NS.

2. Projection identity formalism

The stable caustics of a family of trajectories can be classified in terms of catastrophe theory (Poston and Stewart 1978). On a fine scale, each type of caustic has its characteristic short-wave interference pattern which can be mapped locally onto the canonical 'diffraction catastrophe' wavefunction $\psi_l(C_j)$. Here l is a label for the catastrophe (fold, cusp, elliptic umbilic, etc) and C_j $(1 \leqslant j \leqslant K)$ denotes the control parameters on which ψ depends (e.g. position coordinates or time); the number K is the codimension of l. The ψ_l are obtained from the generating polynomials $\phi_l(S_k; C_j)$ for the catastrophes, involving state variables S_k $(1 \leqslant k \leqslant n)$ as well as the C_j; the number n is the corank of l, and satisfies $n \leqslant 2$ if $K \leqslant 5$. In terms of ϕ_l, the lth diffraction catastrophe has the integral representation (Berry 1976, Duistermaat 1974, Guillemin and Sternberg 1977)

$$\psi_l(C_j) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}^n S_k \, \mathrm{e}^{\mathrm{i}\phi_l(S_k; C_j)}. \tag{3}$$

These diffraction catastrophes generalise the Airy function of equation (1), which corresponds to the simplest case where $K = n = 1$ ('fold' catastrophe).

In deriving the projection identities, the first step is to write the wave intensity as

$$|\psi_l(C_j)|^2 = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}^n S_k \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}^n S'_k \, \mathrm{e}^{\mathrm{i}(\phi_l(S_k; C_j) - \phi_l(S'_k; C_j))}. \tag{4}$$

Under the transformation

$$S_k = u_k + v_k, \qquad S'_k = u_k - v_k, \tag{5}$$

which has Jacobian 2^n, this can be written in either of two forms, depending on the order in which the integrations over u_k and v_k are performed. We introduce the notation

$$\Phi_l(u_k, v_k; C_j) \equiv \phi_l(u_k + v_k; C_j) - \phi_l(u_k - v_k; C_j), \tag{6}$$

and note that Φ_l is an odd function of v_k.

Then the two forms for $|\psi_l|^2$ can be written as follows:

$$|\psi_l(C_j)|^2 = \left(\frac{2}{\pi}\right)^{n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}^n u_k \left\{ \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}^n v_k \, \mathrm{e}^{\mathrm{i}\Phi_l(u_k, v_k; C_j)} \right\} \tag{7}$$

and

$$|\psi_l(C_j)|^2 = \left(\frac{2}{\pi}\right)^{n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}^n v_k \left\{ \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}^n u_k \, \mathrm{e}^{\mathrm{i}\Phi_l(u_k, v_k; C_j)} \right\}. \tag{8}$$

The two sets of identities, which we shall call the 'u' and 'v' identities, are determined by equations (7) and (8) respectively. To derive them, it is necessary to evaluate the functions $\{\ldots\}$, using explicit forms for the generating polynomials ϕ_l. This is a straightforward but tedious exercise, which, however, has the remarkable result that the functions $\{\ldots\}$ can be expressed in terms of particular sections through the diffraction catastrophe ψ_l, or in terms of a diffraction catastrophe of lower codimension, or in terms of powers and exponentials of u_k or v_k.

3. Cuspoids

These have corank $n = 1$, and hence a single state variable S, and can be labelled by their codimension K. The generating polynomials are

$$\phi_K(S; C_j) = \frac{S^{K+2}}{K+2} + \sum_{j=1}^{K} \frac{C_j S^j}{j}. \tag{9}$$

Both the u and v identities take different forms when K is even or odd.

When K *is odd*, the u identities are

$$|\psi_K(C_j)|^2 = \sqrt{\frac{2}{\pi}} \frac{1}{2^{1/(K+2)}} \int_{-\infty}^{\infty} \mathrm{d}u\, \psi_K[\xi_i(u; C_j)], \tag{10}$$

where ψ_K is the diffraction catastrophe (3) with ϕ given by (9), and

$$\begin{aligned} \xi_{2m+1} &= \frac{2^{(K+1-2m)/(K+2)}}{(2m)!}\left(\frac{(K+1)!}{(K+1-2m)!}u^{K+1-2m} + \sum_{j=2m+1}^{K} \frac{C_j (j-1)! u^{j-1-2m}}{(j-1-2m)!}\right) \\ &\quad \text{for } 0 \leqslant m \leqslant \tfrac{1}{2}(K-1), \\ \xi_{2m+2} &= 0 \qquad \text{for } 0 \leqslant m \leqslant \tfrac{1}{2}(K-3). \end{aligned} \tag{11}$$

When K *is even*, the range of integration in the u identities must be reduced to $u \geqslant 0$ so that the fractional powers of u are well defined, which leads to a distinction between odd and even control parameters C_j:

$$|\psi_K(C_j)|^2 = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\mathrm{d}u}{[2u(K+1)]^{1/(K+1)}} (\psi_{K-1}[\xi_i(u; C_j)] + \psi_{K-1}[\xi_i(u; (-1)^j C_j)]), \tag{12}$$

where

$$\begin{aligned} \xi_{2m+1} &= \frac{2}{[2u(K+1)]^{(2m+1)/(K+1)}(2m)!}\left(\frac{(K+1)!}{(K+1-2m)!}u^{K+1-2m}\right. \\ &\quad \left. + \sum_{j=2m+1}^{K} \frac{C_j (j-1)!}{(j-1-2m)!} u^{j-1-2m}\right), \qquad \text{for } 0 \leqslant m \leqslant \tfrac{1}{2}(K-2), \\ \xi_{2m+2} &= 0 \qquad \text{for } 0 \leqslant m \leqslant \tfrac{1}{2}(K-4). \end{aligned} \tag{13}$$

All the v identities require that the range of integration be reduced to $v \geqslant 0$. This is facilitated by the v antisymmetry of $\Phi_K(u, v; C_j)$ and gives

$$|\psi_K(C_j)|^2 = 2\sqrt{\frac{2}{\pi}} \operatorname{Re} \int_0^{\infty} \frac{\mathrm{d}v}{[2v(K+1)]^{1/(K+1)}} \mathrm{e}^{\mathrm{i}\alpha_K(v; C_j)} \psi_{K-1}[\xi_i(v; C_j)], \tag{14}$$

where for K *odd*,

$$\alpha_K = 2\left(\frac{v^{K+2}}{K+2} + \sum_{r=0}^{(K-1)/2} \frac{C_{2r+1}}{2r+1} v^{2r+1}\right), \tag{15}$$

$$\begin{aligned} \xi_{2m+1} &= \frac{2}{[2v(K+1)]^{(2m+1)/(K+1)}(2m)!} \sum_{r=m}^{(K-3)/2} \frac{C_{2r+2}(2r+1)!}{[2(r-m)+1]!} v^{2(r-m)+1} \\ &\quad \text{for } 0 \leqslant m \leqslant \tfrac{1}{2}(K-3), \end{aligned} \tag{16}$$

$$\xi_{2m+2}=\frac{2}{[2v(K+1)]^{(2m+2)/(K+1)}(2m+1)!}\left(\frac{(K+1)!}{(K-2m)!}v^{K-2m}\right.$$
$$\left.+\sum_{r=m}^{(K-3)/2}\frac{C_{2r+3}(2r+2)!}{[2(r-m)+1]!}v^{2(r-m)+1}\right)\qquad\text{for } 0\leqslant m\leqslant\tfrac{1}{2}(K-3),$$

and for K *even*,

$$\alpha_K=2\sum_{r=0}^{(K-2)/2}\frac{C_{2r+1}}{2r+1}v^{2r+1},\tag{17}$$

$$\xi_{2m+1}=\frac{2}{[2v(K+1)]^{(2m+1)/(K+1)}(2m)!}\left(\frac{(K+1)!}{(K+1-2m)!}v^{K+1-2m}\right.$$
$$\left.+\sum_{r=m}^{(K-2)/2}\frac{C_{2r+2}(2r+1)!}{[2(r-m)+1]!}v^{2(r-m)+1}\right)\qquad\text{for } 0\leqslant m\leqslant\tfrac{1}{2}(K-2),$$
$$\xi_{2m+2}=\frac{2}{[2v(K+1)]^{(2m+2)/(K+1)}(2m+1)!}\sum_{r=m}^{(K-4)/2}\frac{C_{2r+3}(2r+2)!}{[2(r-m)+1]!}v^{2(r-m)+1}\tag{18}$$
$$\text{for } 0\leqslant m\leqslant\tfrac{1}{2}(K-4).$$

To illustrate these general results we write explicit formulae for $K=1$ (fold), $K=2$ (cusp) and $K=3$ (swallowtail). For $K=1$, the u and v identities are

$$|\psi_{\text{fold}}(C_1)|^2=\sqrt{\frac{2}{\pi}}2^{-1/3}\int_{-\infty}^{\infty}\mathrm{d}u\,\psi_{\text{fold}}[2^{2/3}(u^2+C_1)]\tag{19}$$

and

$$|\psi_{\text{fold}}(C_1)|^2=2\sqrt{\frac{2}{\pi}}\int_0^{\infty}\mathrm{d}v\,(4v)^{-1/2}\cos\left(\tfrac{2}{3}v^3+2C_1v+\tfrac{1}{4}\pi\right).\tag{20}$$

These relations involve ψ_{fold}, which is simply $(2\pi)^{1/2}$ times the Airy function (1). The u identity (19) is the original relation (2) which we are generalising. In deriving (20) we used the 'zero-order' (non-catastrophic) extrapolation of (3) and (9) in the form

$$\psi_0=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}S\,\mathrm{e}^{\mathrm{i}S^2/2}=\mathrm{e}^{\mathrm{i}\pi/4}.\tag{21}$$

For $K=2$, the identities, involving the function first studied by Pearcey (1946), are

$$|\psi_{\text{cusp}}(C_1,C_2)|^2=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\mathrm{d}u\,(6u)^{-1/3}(\psi_{\text{fold}}[2(6u)^{-1/3}(u^3+C_2u+C_1)]$$
$$+\psi_{\text{fold}}[2(6u)^{-1/3}(u^3+C_2u-C_1)])\tag{22}$$

and

$$|\psi_{\text{cusp}}(C_1,C_2)|^2=2\sqrt{\frac{2}{\pi}}\,\mathrm{Re}\int_0^{\infty}\mathrm{d}v\,(6v)^{-1/3}\,\mathrm{e}^{\mathrm{i}2C_1v}\psi_{\text{fold}}[2(6v)^{-1/3}(v^3+C_2v)].\tag{23}$$

For $K=3$,

$$|\psi_{\text{swallowtail}}(C_1,C_2,C_3)|^2=\sqrt{\frac{2}{\pi}}2^{-1/5}\int_{-\infty}^{\infty}\mathrm{d}u$$
$$\times\psi_{\text{swallowtail}}[2^{4/5}(u^4+C_3u^2+C_2u+C_1),0,2^{2/5}(6u^2+C_3)]\tag{24}$$

and

$$|\psi_{\text{swallowtail}}(C_1, C_2, C_3)|^2 = 2\sqrt{\frac{2}{\pi}}\,\text{Re}\int_0^\infty \mathrm{d}v\,(8v)^{-1/4}\,\mathrm{e}^{\mathrm{i}(2v^5/5+2C_3v^3/3+2C_1v)} \times \psi_{\text{cusp}}[2(8v)^{-1/4}C_2 v,\,4(8v)^{-1/2}(2v^3+C_3v)]. \tag{25}$$

4. Umbilics

These have corank $n = 2$, and hence two state variables S_1 and S_2. We present results only for the first two cases, which both have codimension $K = 3$, namely the elliptic (E) and hyperbolic (H) umbilics.

For the *elliptic umbilic*, the generating function is

$$\phi_{\text{E}}(S_1, S_2; C_1, C_2, C_3) = S_1^3 - 3S_1S_2^2 - C_3(S_1^2+S_2^2) - C_1S_1 - C_2S_2, \tag{26}$$

and the diffraction catastrophe is the function $\psi_{\text{E}}(C_1, C_2, C_3)$ studied in detail by Berry *et al* (1979). The u identity is

$$|\psi_{\text{E}}(C_1, C_2, C_3)|^2 = \frac{2^{1/3}}{\pi}\int_{-\infty}^\infty \mathrm{d}u_1 \int_{-\infty}^\infty \mathrm{d}u_2\,\psi_{\text{E}}\{2^{2/3}[C_1 + 2C_3u_1 + 3(u_2^2 - u_1^2)],\,2^{2/3}(C_2 + 2C_3u_2 + 6u_1u_2),\,0\}. \tag{27}$$

This can be rewritten in terms of Ai and the irregular Airy function Bi (Abramowitz and Stegun 1964) by using the relation (Berry *et al* 1979, Trinkaus and Drepper 1977)

$$\psi_{\text{E}}(A, B, 0) = \left(\frac{2}{3}\right)^{2/3}\pi\,\text{Re}\left[\text{Ai}\left(\frac{-A-\mathrm{i}B}{12^{1/3}}\right)\text{Bi}\left(\frac{-A+\mathrm{i}B}{12^{1/3}}\right)\right], \tag{28}$$

which gives

$$|\psi_{\text{E}}(C_1, C_2, C_3)|^2 = \frac{2}{3^{1/3}}\,\text{Re}\int_0^\infty u\,\mathrm{d}u\int_0^{2\pi}\mathrm{d}\theta\,\text{Ai}\,[3^{-1/3}(-C_1 - \mathrm{i}C_2 - 2C_3u\,\mathrm{e}^{\mathrm{i}\theta} + 3u^2\,\mathrm{e}^{-2\mathrm{i}\theta})] \times \text{Bi}\,[3^{-1/3}(-C_1 + \mathrm{i}C_2 - 2C_3u\,\mathrm{e}^{-\mathrm{i}\theta} + 3u^2\,\mathrm{e}^{2\mathrm{i}\theta})]. \tag{29}$$

The v identity is

$$|\psi_{\text{E}}(C_1, C_2, C_3)|^2 = \frac{1}{3\pi}\int_0^\infty \mathrm{d}v\int_0^\pi \mathrm{d}\theta\,\cos[2v^3\cos 3\theta - 2v(\tfrac{1}{3}C_3^2\cos 3\theta + C_1\cos\theta + C_2\sin\theta)]. \tag{30}$$

For the *hyperbolic umbilic*, the generating function is

$$\phi_{\text{H}}(S_1, S_2; C_1, C_2, C_3) = S_1^3 + S_2^3 + C_3S_1S_2 - C_1S_1 - C_2S_2. \tag{31}$$

The u identity is

$$|\psi_{\text{H}}(C_1, C_2, C_3)|^2 = \frac{2^{1/3}}{\pi}\int_{-\infty}^\infty \mathrm{d}u_1 \times \int_{-\infty}^\infty \mathrm{d}u_2\,\psi_{\text{H}}[2^{2/3}(C_1 - C_3u_2 - 3u_1^2),\,2^{2/3}(C_2 - C_3u_1 - 3u_2^2),\,0]. \tag{32}$$

This can be rewritten in terms of Airy functions using the easily derived relation

$$\psi_H(A, B, 0) = (2\pi/3^{2/3})\,\mathrm{Ai}\,(-3^{-1/3}A)\,\mathrm{Ai}\,(-3^{-1/3}B), \tag{33}$$

which gives

$$|\psi_H(C_1, C_2, C_3)|^2 = (\tfrac{16}{9})^{1/3} \int_{-\infty}^{\infty} \mathrm{d}u_1 \times \int_{-\infty}^{\infty} \mathrm{d}u_2\, \mathrm{Ai}[(\tfrac{4}{3})^{1/3}(3u_1^2 + C_3u_2 - C_1)]\,\mathrm{Ai}[(\tfrac{4}{3})^{1/3}(3u_2^2 + C_3u_1 - C_2)]. \tag{34}$$

Finally, the v identity is

$$|\psi_H(C_1, C_2, C_3)|^2 = \frac{2^{3/2}}{3\pi}\int_0^{\infty} \mathrm{d}v \int_0^{\pi/2} \frac{\mathrm{d}\theta}{(\sin 2\theta)^{1/2}} \times \cos[2v^3\cos^3\theta - v\cos\theta(2C_1 + \tfrac{1}{6}C_3^2\tan^2\theta) + \tfrac{1}{4}\pi] \times \cos[2v^3\sin^3\theta - v\sin\theta(2C_2 + \tfrac{1}{6}C_3^2\cot^2\theta) + \tfrac{1}{4}\pi]. \tag{35}$$

5. Phase-space interpretation

Consider a family of classical trajectories in an n-dimensional coordinate space $\boldsymbol{q}$, represented by a smooth n-dimensional manifold $\mathscr{C}$ in the $2n$-dimensional phase space $\boldsymbol{q}, \boldsymbol{p}$. We are interested in families of trajectories with caustics, so that there are several momenta $\boldsymbol{p}_i(\boldsymbol{q})$ for each $\boldsymbol{q}$, and $\mathscr{C}$ is folded over $\boldsymbol{q}$ (figure 1).

Associated with $\mathscr{C}$ is a semiclassical quantum state whose coordinate wavefunction will be denoted by $\psi(\boldsymbol{q})$. A 'WKB' approximation to $\psi(\boldsymbol{q})$, constructed out of local plane-wave contributions from each $\boldsymbol{p}_i(\boldsymbol{q})$, fails on caustics, where two or more $\boldsymbol{p}_i$ coalesce and the contributions diverge. It was realised by Maslov (1972) that, because

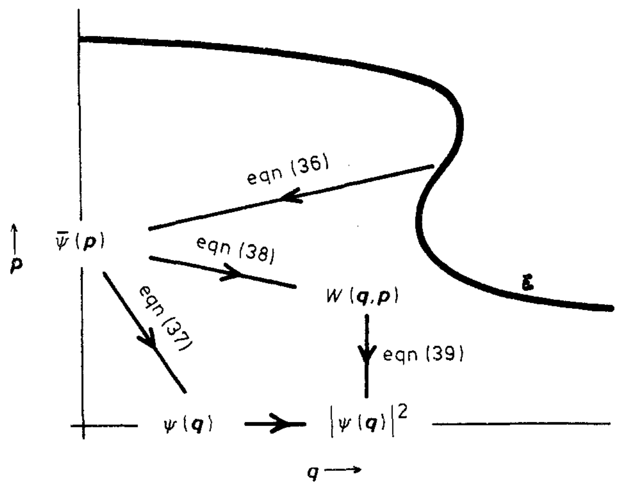

Figure 1. Manifold $\mathscr{C}$ of classical trajectories, leading to formulae for the coordinate and momentum wavefunctions and the Wigner function.

$\mathscr{C}$ is smooth, those regions giving caustics in $\boldsymbol{q}$ cannot be folded over $\boldsymbol{p}$. Therefore $\boldsymbol{q}$ is locally a single-valued function of $\boldsymbol{p}$, and the WKB method can be used to give a satisfactory approximation to the *momentum* wavefunction $\bar{\psi}(\boldsymbol{p})$. In the case where trajectories on $\mathscr{C}$ are distributed uniformly in momentum, this approximation is

$$\bar{\psi}(\boldsymbol{p}) = A \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{\boldsymbol{p}_0}^{\boldsymbol{p}} \boldsymbol{q}(\boldsymbol{p}') \cdot \mathrm{d}\boldsymbol{p}'\right), \tag{36}$$

where A and $\boldsymbol{p}_0$ are constant. The desired approximate wavefunction $\psi(\boldsymbol{q})$ can now be obtained by a Fourier transformation:

$$\psi(\boldsymbol{q}) = Ah^{-n/2}\int \mathrm{d}\boldsymbol{p} \exp\left[\frac{\mathrm{i}}{\hbar}\left(-\int_{\boldsymbol{p}_0}^{\boldsymbol{p}} \boldsymbol{q}(\boldsymbol{p}') \cdot \mathrm{d}\boldsymbol{p}' + \boldsymbol{q} \cdot \boldsymbol{p}\right)\right]. \tag{37}$$

An alternative procedure is to employ $\bar{\psi}(\boldsymbol{p})$ to construct the phase-space quantum function of Wigner (1932). This is

$$W(\boldsymbol{q}, \boldsymbol{p}) \equiv (\pi\hbar)^{-n}\int \mathrm{d}\boldsymbol{P} \exp\left(\frac{2\mathrm{i}}{\hbar}\boldsymbol{P} \cdot \boldsymbol{q}\right)\bar{\psi}(\boldsymbol{p}+\boldsymbol{P})\bar{\psi}^*(\boldsymbol{p}-\boldsymbol{P}). \tag{38}$$

Wigner's function has the property that under projection along $\boldsymbol{p}$ it gives the coordinate probability density

$$|\psi(\boldsymbol{q})|^2 = \int \mathrm{d}\boldsymbol{p}\, W(\boldsymbol{q}, \boldsymbol{p}). \tag{39}$$

Using approximation (37) for $\psi(\boldsymbol{q})$, and approximations (36) and (38) for $W(\boldsymbol{q}, \boldsymbol{p})$, the Wigner projection identity becomes

$$\left|h^{-n/2}\int \mathrm{d}\boldsymbol{p} \exp\left[\frac{\mathrm{i}}{\hbar}\left(-\int_{\boldsymbol{p}_0}^{\boldsymbol{p}} \boldsymbol{q}(\boldsymbol{p}') \cdot \mathrm{d}\boldsymbol{p}' + \boldsymbol{q} \cdot \boldsymbol{p}\right)\right]\right|^2$$
$$= \left(\frac{2}{\pi\hbar}\right)^{n/2}\int \mathrm{d}\boldsymbol{p}\left\{h^{-n/2}\int \mathrm{d}\boldsymbol{P} \exp\left[\frac{\mathrm{i}}{\hbar}\left(-\int_{\boldsymbol{p}-\boldsymbol{P}}^{\boldsymbol{p}+\boldsymbol{P}} \boldsymbol{q}(\boldsymbol{p}') \cdot \mathrm{d}\boldsymbol{p}' + 2\boldsymbol{q} \cdot \boldsymbol{P}\right)\right]\right\}. \tag{40}$$

This is an exact relation between two semiclassical approximations. The 'Maslov' and 'Wigner' routes leading respectively to the left and right members of this equation are shown schematically in figure 1.

Obviously the identity (40) resembles the diffraction catastrophe projection identities (7) and (8). To make the connection precise, we use the fact that the folded parts of $\mathscr{C}$ are locally equivalent under diffeomorphism to the 'critical manifold' (Poston and Stewart 1978) of one of the catastrophes. Therefore the phase in the Maslov wavefunction (37) can be mapped onto the corresponding catastrophe generating polynomial. The components of momentum $\boldsymbol{p}$ then correspond to the state variables. After scaling away $\hbar$ (which is non-trivial—see Varchenko 1976, Berry 1977b), $\psi(\boldsymbol{q})$ becomes one of the diffraction catastrophes (3), with control parameters consisting of the components of $\boldsymbol{q}$ together with any other quantities (e.g. time or energy) on which ψ depends. If the $\boldsymbol{P}$ and $\boldsymbol{p}$ integrations in (40) are carried out in the order indicated, the result is the u identity (7); interchanging the order of integration gives the v identity (8).

6. Wigner catastrophes

It follows from § 5 that the integrand of each u projection identity represents a Wigner function in the phase space whose momentum variables are u (cuspoids) or u_1, u_2 (umbilics) and whose coordinate variables are C_1 (cuspoids) or C_1, C_2 (umbilics), and for which any remaining C_j are extra parameters (e.g. energy or time). As can be seen from equations (10), (12), (19), (22), (24), (27) and (32), these Wigner functions, which project to give the intensities of diffraction catastrophes, are themselves given by particular hypersections (i.e. restrictions to lower-dimensional subsets of control space) of particular diffraction catastrophes (uniquely up to change of integration variable); therefore we call them 'Wigner catastrophes'. Because Wigner functions must be real (equation (38)), the hypersections must be those on which the diffraction catastrophes are real, implying that only those diffraction catastrophes possessing real hypersections can occur as Wigner catastrophes.

First we consider the *cuspoids*. The diffraction catastrophe intensities for cuspoids of codimension $K=2m+1$ and $K=2m+2$ are both generated by projecting the Wigner catastrophe with $K=2m+1$. This is because only cuspoid diffraction catastrophes of odd codimension possess a real hypersection, obtained by setting all even controls C_{2m} equal to zero. This real hypersection intersects the diffraction catastrophe caustic in several branches, on which an even number of stationary phase points (SPPs) of the integrand of (3) coalesce. The form (9) of the generating polynomial ϕ_K shows that there is always one and only one branch, given by $C_1=0$, on every point of which (apart from a set of zero measure) just one pair of SPPs coalesces, corresponding locally to a simple fold catastrophe. In addition there may also be branches on which SPPs coalesce in two or more separate pairs, corresponding to self-intersections of the caustic. Finally, there may be branches on which four or more SPPs coalesce, corresponding locally to swallowtail or higher catastrophes.

Equations (11) and (13) define maps M between the canonical controls ξ_i of the catastrophes, and the phase-space variables u, C_1 and extra parameters $C_{j\geq 2}$. These map real hypersections of diffraction catastrophes onto Wigner catastrophes. Berry (1977a) showed that, on the classical phase-space manifold $\mathscr{C}$ of a system with one degree of freedom (such as we are considering here), a single pair of SPPs coalesces. This shows that $\mathscr{C}$ is the image under M of the fold caustic of the diffraction catastrophe; the equation of this fold in canonical coordinates is $\xi_1(u;C_j)=0$. Then both (11) and (13) imply that its image in phase space has the equation

$$\partial\phi_K(u;C_j)/\partial u=0, \tag{41}$$

where $|\psi_K(C_j)|^2$ is the diffraction catastrophe onto which the Wigner catastrophe projects. Therefore the projection identities generate Wigner functions whose classical manifolds are exactly the catastrophe manifolds of the $\psi_K(C_j)$.

The other branches of the caustics in the real hypersections, where two or more separate pairs of SPPs coalesce, map into those lines of the Wigner catastrophes that were labelled $\mathscr{L}$ by Berry (1977a). On $\mathscr{L}$, the strength of the Wigner function is high but its sign alternates, so that its projection is weak and is not a caustic of $\psi_K(C_j)$. This behaviour constrasts with $\mathscr{C}$, on which the sign of the Wigner function remains positive and which does project onto a caustic of $\psi_K(C_j)$.

Now we consider in detail equations (19), (22) and (24) which give the Wigner catastrophes $W(C_1,u;C_{j\geq 2})$ projecting onto the fold, cusp and swallowtail respectively. In the first case, equation (19) shows that the fold diffraction intensity is projected from

$$W(C_1, u) = (2^{1/6}/\sqrt{\pi})\psi_{\text{fold}}[2^{2/3}(u^2 + C_1)]. \tag{42}$$

This is another fold diffraction function whose classical manifold $\mathscr{C}$, given by $\xi_1(u; C_1) = 0$, is the parabola $u^2 + C_1 = 0$; there are parabolic Airy fringes on the concave side of $\mathscr{C}$.

The Wigner catastrophe that projects onto a cusp is obtained by the transformation of (22) as

$$W(C_1, u; C_2) = (2/\pi)^{1/2}(6|u|)^{-1/3}\psi_{\text{fold}}[2 \operatorname{sgn} u(6|u|)^{-1/3}(u^3 + C_2u + C_1)]. \tag{43}$$

The classical manifold $\mathscr{C}$ given by $\xi_1(u; C_1, C_2) = 0$ is $u^3 + C_2u + C_1 = 0$, which is the cusp catastrophe manifold. It is surprising that merely by distorting a fold diffraction catastrophe it can be made to project into the more complicated cusp intensity pattern. There are Airy fringes between $\mathscr{C}$ and $u = 0$ on the concave sides of $\mathscr{C}$. These are sketched in figure 2 for a case where $C_2 < 0$. Approaching the singular line $u = 0$, the fringes tend to zero spacing and infinite strength in such a way that the average value of W is small. This singular behaviour exemplifies those cases where $\mathscr{C}$ is antisymmetric about its inflections (as in equation (5.13) of Berry 1977a), giving rise to a 'catastrophe of infinite order' where the line $\mathscr{L}$ condenses onto a single point ($u = C_1 = 0$ in this case) on $\mathscr{C}$.

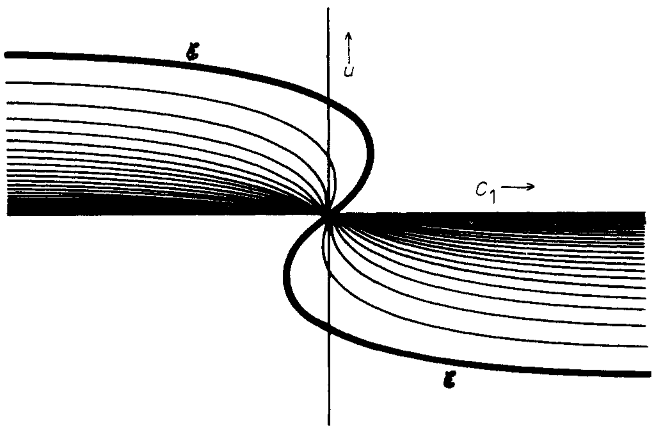

Figure 2. Fringes (thin lines) of the Wigner catastrophe around a cubic classical manifold $\mathscr{C}$ (bold curve) for $C_2 < 0$.

The Wigner catastrophe that projects onto a swallowtail is, from (24),

$$W(C_1, u; C_2, C_3) = (2/\pi)^{1/2}2^{-1/5} \times \psi_{\text{swallowtail}}[2^{4/5}(u^4 + C_3u^2 + C_2u + C_2), 0, 2^{2/5}(6u^2 + C_3)]. \tag{44}$$

This involves the real hypersection $\psi_{\text{swallowtail}}(\xi_1, 0, \xi_3)$ whose caustic, shown in figure 3, has two branches.

On the first branch, $\xi_1 = 0$, a single pair of SPPs coalesces. This maps under M onto the classical manifold $\mathscr{C}$ given by

$$u^4 + C_3u^2 + C_2u + C_1 = 0, \tag{45}$$

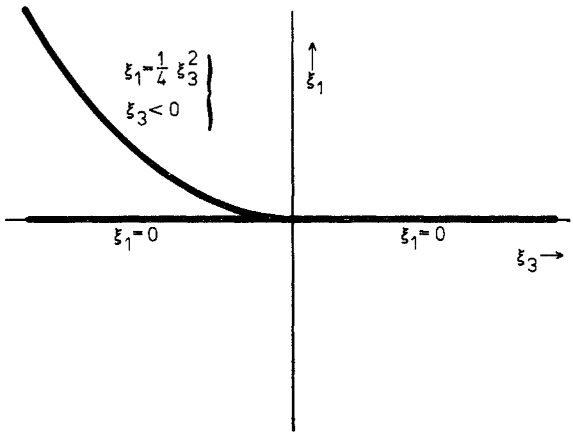

Figure 3. The real section ($\xi_2 = 0$) through the swallowtail caustic in canonical coordinates (ξ_1, ξ_2, ξ_3).

which is exactly the swallowtail catastrophe manifold. On the second branch, whose equation is

$$\xi_1 = \tfrac{1}{4}\xi_3^2, \qquad \xi_3 < 0, \tag{46}$$

the SPPS coalesce in two separate pairs. Under M this branch (which can occur only if $C_3 < 0$) maps onto the line $\mathscr{L}$ of Berry (1977a), given by

$$32u^4 + 8C_3u^2 - 4C_2u + C_3^2 - 4C_1 = 0, \qquad 6u^2 + C_3 < 0. \tag{47}$$

This line $\mathscr{L}$ joins the two inflections (swallowtail points) of $\mathscr{C}$. If $C_2 = 0$, $\mathscr{L}$ is symmetric (as shown on figure 4 of Berry (1977a) with p and q interchanged). An asymmetric version with $C_2 > 0$ is shown in figure 4; points near the three maxima of $\mathscr{C}$ and $\mathscr{L}$ as a function of C_1 always share a common tangent, as they must by the chord construction of Berry (1977a).

Analysis of the Wigner catastrophes projecting into the higher cuspoids is complicated by the large number of controls $C_{j\geq 2}$ in addition to the phase-space variables C_1, u. We mention only the fact that beyond the swallowtail the next diffraction catastrophe with a real hypersection has codimension five and organises six SPPS, suggesting that this catastrophe could describe the Wigner function near cusps of $\mathscr{L}$ (as illustrated in figure 2 of Berry (1977a)).

Finally, we briefly consider the *umbilics*, which can occur only in systems with at least two degrees of freedom. According to equation (27), the Wigner catastrophe projecting onto the *elliptic* umbilic is the real section $\xi_3 = 0$ of the elliptic umbilic diffraction catastrophe itself. In this section the caustic is the single point $\xi_1 = \xi_2 = 0$ at which all four SPPS coalesce. This point maps under M into a classical 2-manifold (labelled $\mathscr{T}$ by Berry 1977a) in the four-dimensional phase space with variables C_1, C_2, u_1, u_2 (together with C_3 as extra parameter). $\mathscr{T}$ has equations

$$3(u_1^2 - u_2^2) - 2C_3u_1 - C_1 = 0, \qquad -6u_1u_2 - 2C_3u_2 - C_2 = 0, \tag{48}$$

corresponding precisely to the elliptic umbilic catastrophe manifold. In this Wigner catastrophe, $\mathscr{T}$ is decorated by Airy-like fringes (cf equation (28)), and there are no $\mathscr{L}$-type branches.

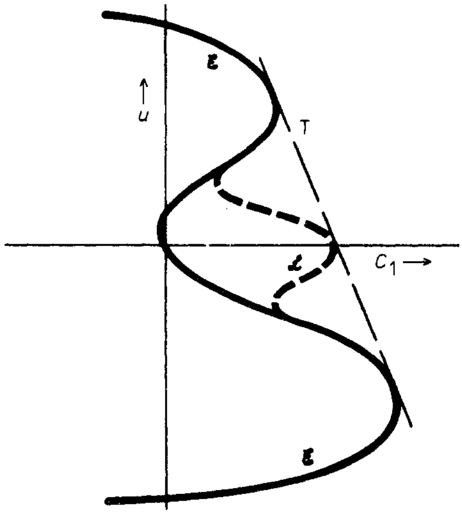

Figure 4. The classical manifold $\mathscr{C}$ and catastrophe line $\mathscr{L}$ (broken curve) of a swallowtail Wigner catastrophe, for $C_2 > 0$, $C_3 < 0$. 'T' is the common tangent with equation $C_1 = \frac{1}{4}C_3^2 - C_2 u$.

According to equation (29), the Wigner catastrophe projecting onto the *hyperbolic* umbilic is the real section $\xi_3 = 0$ of the hyperbolic umbilic diffraction catastrophe itself. In this section the caustic consists of the single point $\xi_1 = \xi_2 = 0$ at which all four SPPs coalesce, together with the two half-lines $\xi_1 = 0$, $\xi_2 > 0$ and $\xi_2 = 0$, $\xi_1 > 0$ on which the SPPs coalesce in two separate pairs. The point $\xi_1 = \xi_2 = 0$ maps under M into the classical manifold $\mathscr{T}$ identical to the hyperbolic umbilic catastrophe manifold, again decorated with Airy fringes (cf equation (33)). The two half-lines map into two three-dimensional regions in phase space (for fixed C_3), which are the generalisations of the catastrophe lines $\mathscr{L}$ for the cuspoids.

7. Concluding remarks

The projection identities advance our rather meagre knowledge of the hierarchy of new wavefunctions represented by the diffraction catastrophes (3), and also of the corresponding Wigner catastrophes in phase space. They show how the wavefunction density $|\psi_l|^2$, near a caustic equivalent to a catastrophe labelled l, is determined by a Wigner function which is a distorted version of either the wavefunction $\psi_{l'}$, corresponding to a less singular catastrophe l', or a special section through the wavefunction ψ_l itself.

One way of generalising our analysis would be to employ a general rotation of axes in the S_k, S'_k space, instead of the special rotation in equation (3). This would generate continuous families of projection identities, but preliminary study shows that these are very complicated, even for the simplest catastrophes.

Another extension would be to calculate the identities for the higher catastrophes on the list of Arnol'd (1975). The integrals for those catastrophes involving 'modality' might present difficult convergence problems.

Acknowledgment

FJW wishes to acknowledge financial support by the Science Research Council.

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